Runge-Kutta Methods

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Second Order Runge-Kutta Method

$$\frac{dy}{dx} = f(x, y) \quad \text{with the initial condition: } y(x_1) = y_1$$

The general form of second-order Runge-Kutta methods is:

$$y_{i+1} = y_i + (c_1 K_1 + c_2 K_2)h$$

Con:

$$K_1 = f(x_i, y_i)$$

$$K_2 = f(x_i + a_2 h, y_i + b_{21} K_1 h)$$

where c_1, c_2, a_2 , and b_{21} are constants. The values of these constants vary with the specific second-order method.

Modified Euler method in the form of a second-order Runge-Kutta method

For the modified Euler method, the constants are:

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad a_2 = 1, \quad \text{and} \quad b_{21} = 1$$

Substituting these constants yields:

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)h$$

$$K_1 = f(x_i, y_i)$$
$$K_2 = f(x_i + h, y_i + K_1 h)$$

Midpoint method in the form of a second-order Runge-Kutta method

For the midpoint method, the constants are:

$$c_1 = 0, \quad c_2 = 1, \quad a_2 = \frac{1}{2}, \quad \text{and} \quad b_{21} = \frac{1}{2}$$

Substituting these constants yields:

$$y_{i+1} = y_i + K_2 h$$

$$K_1 = f(x_i, y_i)$$

$$K_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h\right)$$

Heun's method

In Heun's method the constants are:

$$c_1 = \frac{1}{4}, \quad c_2 = \frac{3}{4}, \quad a_2 = \frac{2}{3}, \quad \text{and} \quad b_{21} = \frac{2}{3}$$

Substituting these constants yields:

$$y_{i+1} = y_i + \left(\frac{1}{4}K_1 + \frac{3}{4}K_2\right)h$$

$$K_1 = f(x_i, y_i)$$

$$K_2 = f\left(x_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1 h\right)$$

Example 1

Second Order Runge-Kutta Method in form of modified Euler

Problem:

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4$$

Use RK2 to find $y(1.01), y(1.02)$

$h = 0.01$

$f(x, y) = 1 + y^2 + x^3$

$x_0 = 1, \quad y_0 = -4$

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)h$$

$$K_1 = f(x_i, y_i)$$

$$K_2 = f(x_i + h, y_i + K_1 h)$$

Step 1:

$$K_1 = f(x_0, y_0) = (1 + y_0^{\ 2} + x_0^{\ 3}) = 18.0$$

$$K_2 = f(x_0 + h, y_0 + K_1 h) = (1 + (y_0 + 0.18)^2 + (x_0 + .01)^3) = 16.92$$

$$y_1 = y_0 + \frac{h}{2}\left(K_1 + K_2\right) = -4 + \frac{0.01}{2}(18 + 16.92) = -3.8254$$

Example 1

Second Order Runge-Kutta Method: Modified Euler

Problem:

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4$$

$$Use\ RK2\ to\ find\ y(1.01), y(1.02)$$

$$\text{h} = 0.01$$

$$f(x, y) = 1 + y^2 + x^3$$

$$x_1 = 1.01, \quad y_1 = -3.8254$$

Step 2:

$$K_1 = f(x_1, y_1) = (1 + y_1^{\ 2} + x_1^{\ 3}) = 16.66$$

$$K_2 = f(x_1 + h, y_1 + K_1 h) = (1 + (y_1 + 0.1666)^2 + (x_1 + .01)^3) = 15.45$$

$$y_2 = y_1 + \frac{h}{2}\left(K_1 + K_2\right) = -3.8254 + \frac{0.01}{2}(16.66 + 15.45) = -3.6648$$

Example 1

Summary of the solution

Problem:

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4$$

$Use\ RK2\ to\ find\ y(1.01), y(1.02)$

Summary of the solution

i	x_i	y_i
0	1.00	-4.0000
1	1.01	-3.8254
2	1.02	-3.6648

Numerically Solving ODE in Matlab

Problem:

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4, \text{ over the interval } [1,2].$$

Using Euler modified with h=0.01, 0.02 y 0.5

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)h$$

$$K_1 = f(x_i, y_i)$$
$$K_2 = f(x_i + h, y_i + K_1 h)$$

- Step 1: Create a M-file for dy/dx as firstode.m

```
function yprime=firstode(x,y);
    yprime=1+y^2+x^3;
```

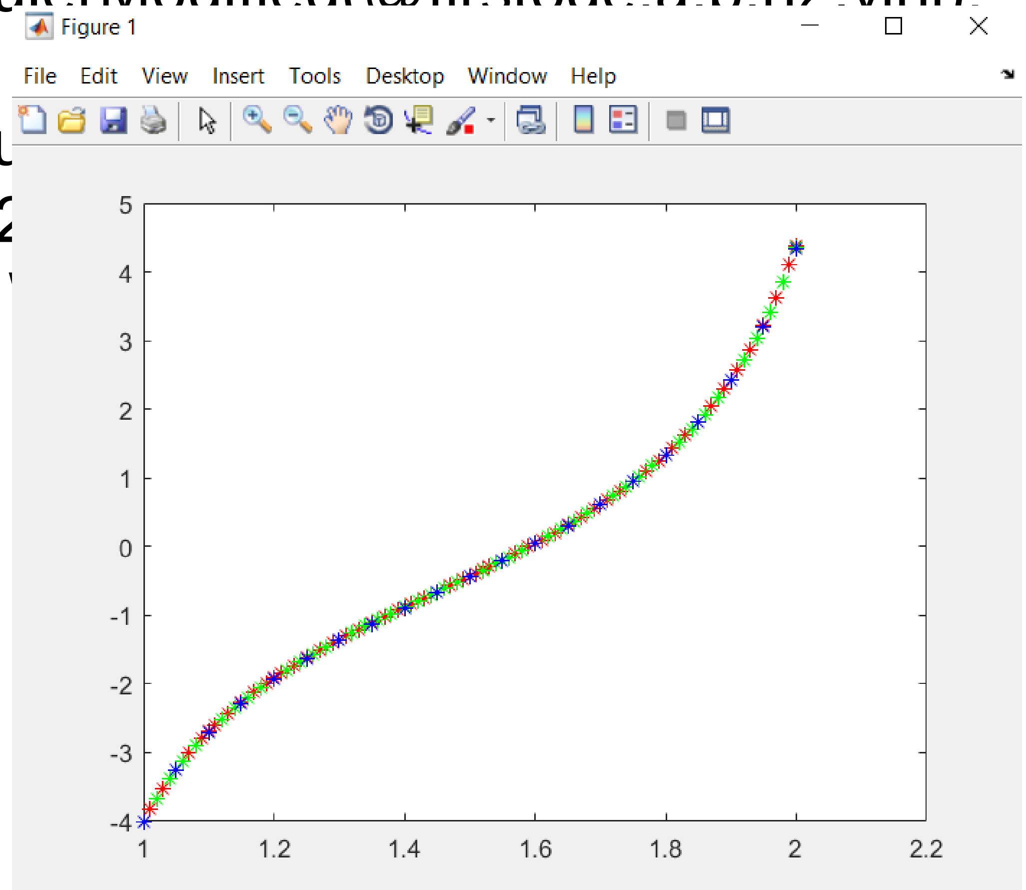
Step 2: Create a M-file to implement Euler modified. The Matlab program must return two column vectors, the first with values of x and the second with value of y.

```
function [ x,y ]=odeRK2EulerModified(ODE,a,b,h,yini)
% Variables de entrada:
% ODE:  Nombre para la función que calcula dy/dx.
% a: Primer valor del intervalo de solucion x.
% b: Ultimo valor del intervalo de solucion x.
% h: Tamaño de paso.
% yini: valor inicial.
% De variables de salida:
% x: Un vector con las coordenadas x de la solución.
% y: Un vector con las coordenadas y de la solución.
x(1) = a; y(1)=yini;
n = (b-a)/h;
for i = 1:n
    x(i+1)=x(i)+h;
    K1 = ODE(x(i),y(i));
    xh = x(i) + h;
    yK1 = y(i) + K1*h;
    K2 = ODE(xh,yK1);
    y (i + 1) =y (i) + (K1+K)*h*0.5;
end

end
```

Step 3: At the Matlab command window

```
>> a = 1; b = 2;
>> h1 = 0.01; yini = -4;
>> [X1,Y1 ]=odeRK2EulerModified(@firstode,a,b,h1,yini);
>> h1 = 0.02;
>> [X2,Y2 ]=odeRK2EulerModified(@firstode,a,b,h2,yini);
>> h1 = 0.5;
>> [X3,Y3 ]=odeRK2Eu
>> plot(X1,Y1,'*r',X2,Y2
>> fprintf('%4.2f %4.6f \
```



Runge-Kutta Methods

Third Order Runge Kutta (RK3)

$$K_1 = f(x_i, y_i)$$

$$K_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h\right)$$

$$K_3 = f\left(x_i + \frac{1}{2}h, y_i - K_1 h + 2K_2 h\right)$$

$$y(x+h) = y(x) + \frac{1}{6}\left(K_1 + 4K_2 + K_3\right)$$

Problem

Consider the following first-order ODE:

$$\frac{dy}{dt} = \frac{y}{t} - 0.5t^2 \qquad \text{from} \quad t = 2 \quad \text{to} \quad t = 5 \quad \text{with} \quad y(2) = 4$$

(a) Solve with the Heun's method using h = 0.5, 1.

(b) Solve with the classical third-order Runge-Kutta method using h = 1.

The analytical solution of the ODE is:

$$y = -\frac{t^3}{4} + 3t$$

In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Classical Fourth-Order Runge-Kutta Method

The most popular RK methods are fourth-order, and the most commonly used form is:

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where:

$$k_1 = f\left(t_i, y_i\right)$$
$$k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$
$$k_3 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$
$$k_4 = f\left(t_i + h, y_i + k_3 h\right)$$

Exercise

Solving by hand a first-order ODE using the fourth-order Runge-Kutta method.

$\frac{dy}{dx} = -1.2y + 7e^{-0.3x}$ from $x = 0$ to $x = 2.5$ with the initial condition $y = 3$ at $x = 0$.

Using $h = 0.5$.

Solution:

The first point of the solution is (0,3), which is the point where the initial condition is given. The values of x and y at the first point are $x_1=0$ and $y_1=3$.
The rest of the solution is done in steps. In each step the next value of the independent variable is calculated by:

$$x_{i+1} = x_i + h = x_i + 0.5$$

The value of the dependent variable Y_{i+1} is calculated by first evaluating K1, K2, K3 and K4 using:

$$K_1 = f(x_i, y_i)$$
$$K_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h\right)$$
$$K_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_2 h\right)$$
$$K_4 = f(x_i + h, y_i + K_3 h)$$

And then substituting the Ks:

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h$$

First step: In the first step i = 1

$$x_2 = x_1 + 0.5 = 0 + 0.5 = 0.5$$

$$K_1 = -1.2y_1 + 7e^{-0.3x_1} = -1.2 \cdot 3 + 7e^{-0.3 \cdot 0} = 3.4$$

$$x_1 + \frac{1}{2}h = 0 + \frac{1}{2} \cdot 0.5 = 0.25 \qquad y_1 + \frac{1}{2}K_1 h = 3 + \frac{1}{2} \cdot 3.4 \cdot 0.5 = 3.85$$

$$K_2 = -1.2\left(y_1 + \frac{1}{2}K_1 h\right) + 7e^{-0.3\left(x_1 + \frac{1}{2}h\right)} = -1.2 \cdot 3.85 + 7e^{-0.3 \cdot 0.25} = 1.874$$

$$y_1 + \frac{1}{2}K_2 h = 3 + \frac{1}{2} \cdot 1.874 \cdot 0.5 = 3.469$$

$$K_3 = -1.2\left(y_1 + \frac{1}{2}K_2 h\right) + 7e^{-0.3\left(x_1 + \frac{1}{2}h\right)} = -1.2 \cdot 3.469 + 7e^{-0.3 \cdot 0.25} = 2.331$$

$$y_1 + K_3 h = 3 + 2.331 \cdot 0.5 = 4.166$$

$$K_4 = -1.2(y_1 + K_3 h) + 7e^{-0.3(x_1 + h)} = -1.2 \cdot 4.166 + 7e^{-0.3 \cdot 0.5} = 1.026$$

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h = 3 + \frac{1}{6}(3.4 + 2 \cdot 1.874 + 2 \cdot 2.331 + 1.026) \cdot 0.5 = 4.069$$

At the end of the first step: $x_2 = 0.5$, $y_2 = 4.069$

Second step: In the second step i = 2

$$x_3 = x_2 + 0.5 = 0.5 + 0.5 = 1.0$$

$$K_1 = -1.2y_2 + 7e^{-0.3x_2} = -1.2 \cdot 4.069 + 7e^{-0.3 \cdot 0.5} = 1.142$$

$$x_2 + \frac{1}{2}h = 0.5 + \frac{1}{2} \cdot 0.5 = 0.75 \qquad y_2 + \frac{1}{2}K_1 h = 4.069 + \frac{1}{2} \cdot 1.142 \cdot 0.5 = 4.355$$

$$K_2 = -1.2\left(y_2 + \frac{1}{2}K_1 h\right) + 7e^{-0.3\left(x_2 + \frac{1}{2}h\right)} = -1.2 \cdot 4.355 + 7e^{-0.3 \cdot 0.75} = 0.3636$$

$$y_2 + \frac{1}{2}K_2 h = 4.069 + \frac{1}{2} \cdot 0.3636 \cdot 0.5 = 4.16$$

$$K_3 = -1.2\left(y_2 + \frac{1}{2}K_2 h\right) + 7e^{-0.3\left(x_2 + \frac{1}{2}h\right)} = -1.2 \cdot 4.16 + 7e^{-0.3 \cdot 0.75} = 0.5976$$

$$y_2 + K_3 h = 4.069 + 0.5976 \cdot 0.5 = 4.368$$

$$K_4 = -1.2(y_2 + K_3 h) + 7e^{-0.3(x_2 + h)} = -1.2 \cdot 4.368 + 7e^{-0.3 \cdot 1.0} = -0.0559$$

$$y_3 = y_2 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h = 4.069 + \frac{1}{6}[1.142 + 2 \cdot 0.3636 + 2 \cdot 0.5976 + (-0.0559)] \cdot 0.5 = 4.32$$

At the end of the second step: $x_3 = 1.0$, $y_3 = 4.32$

Third step: In the third step i = 3

$$x_4 = x_3 + 0.5 = 1.0 + 0.5 = 1.5$$

$$K_1 = -1.2y_3 + 7e^{-0.3x_3} = -1.2 \cdot 4.32 + 7e^{-0.3 \cdot 1.0} = 0.001728$$

$$x_3 + \frac{1}{2}h = 1.0 + \frac{1}{2} \cdot 0.5 = 1.25 \qquad y_3 + \frac{1}{2}K_1 h = 4.32 + \frac{1}{2} \cdot 0.001728 \cdot 0.5 = 4.320$$

$$K_2 = -1.2\left(y_3 + \frac{1}{2}K_1 h\right) + 7e^{-0.3\left(x_3 + \frac{1}{2}h\right)} = -1.2 \cdot 4.32 + 7e^{-0.3 \cdot 1.25} = -0.373$$

$$y_3 + \frac{1}{2}K_2 h = 4.32 + \frac{1}{2} \cdot (-0.373) \cdot 0.5 = 4.227$$

$$K_3 = -1.2\left(y_3 + \frac{1}{2}K_2 h\right) + 7e^{-0.3\left(x_3 + \frac{1}{2}h\right)} = -1.2 \cdot 4.227 + 7e^{-0.3 \cdot 1.25} = -0.2614$$

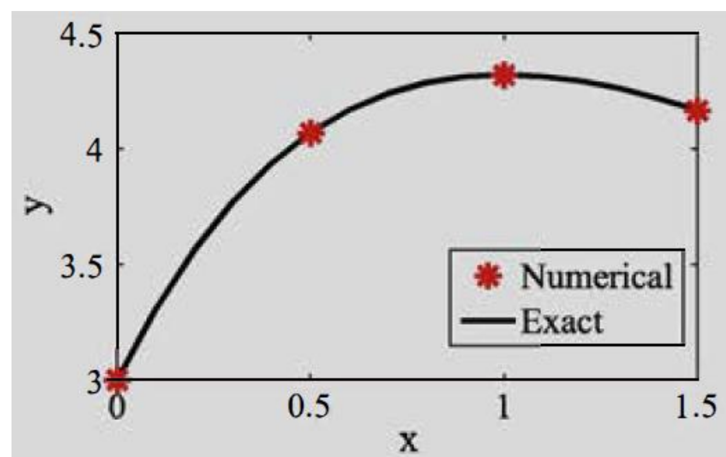
$$y_3 + K_3 h = 4.32 + (-0.2614) \cdot 0.5 = 4.189$$

$$K_4 = -1.2(y_3 + K_3 h) + 7e^{-0.3(x_3 + h)} = -1.2 \cdot 4.189 + 7e^{-0.3 \cdot 1.5} = -0.5634$$

$$y_4 = y_3 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h = 4.32 + \frac{1}{6}[0.001728 + 2 \cdot (-0.373) + 2 \cdot (-0.2614) + (-0.5634)] \cdot 0.5 = 4.167$$

At the end of the third step: $x_4 = 1.5$, $y_4 = 4.167$

i	1	2	3	4
x_i	0.0	0.5	1.0	1.5
y_i (numerical)	3.0	4.069	4.32	4.167



Problem

Write a user-defined MATLAB function that solves a first-order ODE using the classical fourth order Runge-Kutta method.

$\frac{dy}{dx} = -1.2y + 7e^{-0.3x}$ from $x = 0$ to $x = 2.5$ with the initial condition $y = 3$ at $x = 0$.

Compare the results with the exact (analytical) solution: $y = \frac{70}{9}e^{-0.3x} - \frac{43}{9}e^{-1.2x}$.

Using h=0.05.

Solution

To solve the problem, a user-defined MATLAB function called odeRK4, which solves a first-order initial value ODE, is written. The function is then used in a script file, which also generates a plot that shows a comparison between the numerical and the exact solutions. The ODE itself is written in a separate user-defined function that is used by the odeRK4 function.

odeRK4.m | ODEFxy.m | +

```
function [ x,y ]=odeRK4(ODE,a,b,h,yini)
% Variables de entrada:
% ODE:  Nombre para la función que calcula dy/dx.
% a: Primer valor del intervalo de solucion x.
% b: Ultimo valor del intervalo de solucion x.
% h: Tamaño de paso.
% yini: valor inicial.
% De variables de salida:
% x: Un vector con las coordenadas x de la solución.
% y: Un vector con las coordenadas y de la solución.
x(1) = a; y(1)=yini;
n = (b-a)/h
for i = 1:n
    x(i+1)=x(i)+h;
    K1 = ODE(x(i),y(i));
    xhalf = x(i) + h/2;
    yK1 = y(i) + K1*h/2;
    K2 = ODE(xhalf,yK1);
    yK2 = y(i) + K2*h/2;
    K3 = ODE(xhalf,yK2);
    yK3 = y(i) + K3*h;
    K4 = ODE(x(i + 1),yK3);
    y (i + 1) =y (i) + (K1+ 2*K2 + 2*K3 + K4) *h/6;
end

end
```

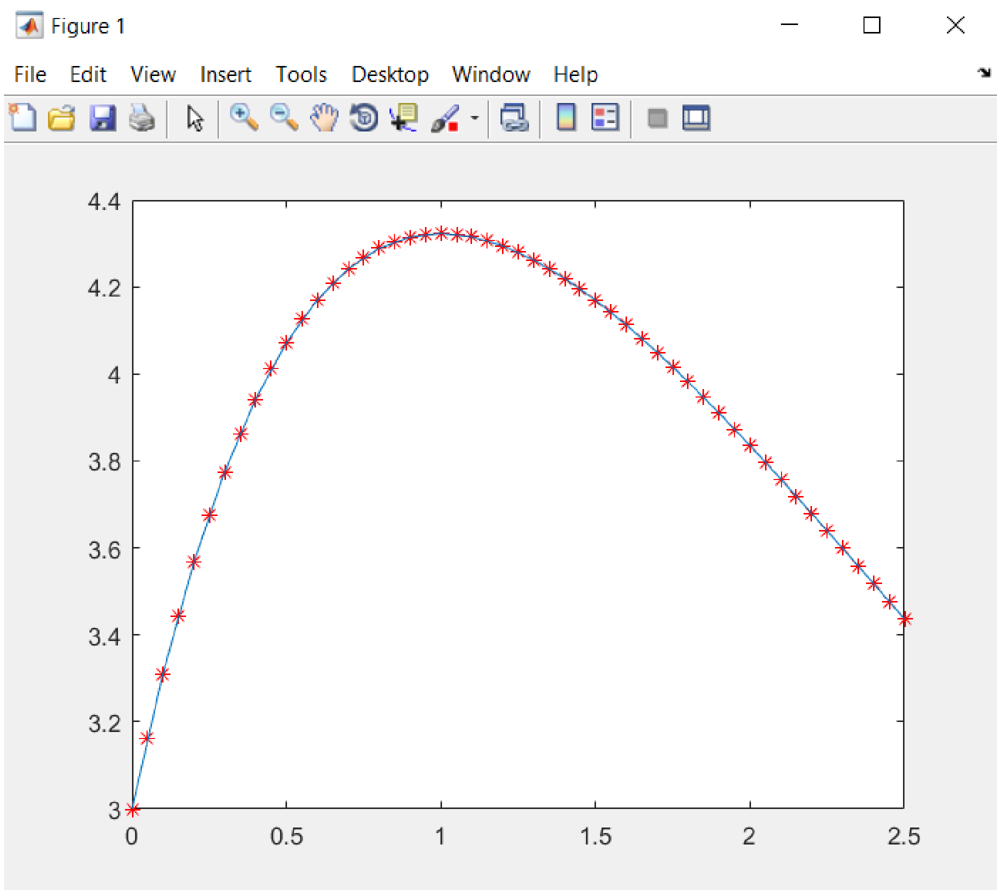
Funcion ODEFxy

odeRK4.m | ODEFxy.m

```matlab
function dydx = ODEFxy(x,y)
%UNTITLED2 Summary of this function goes here
%   Detailed explanation goes here
dydx=-1.2*y+7*exp(-0.3*x);
end
```

At the Matlab command window

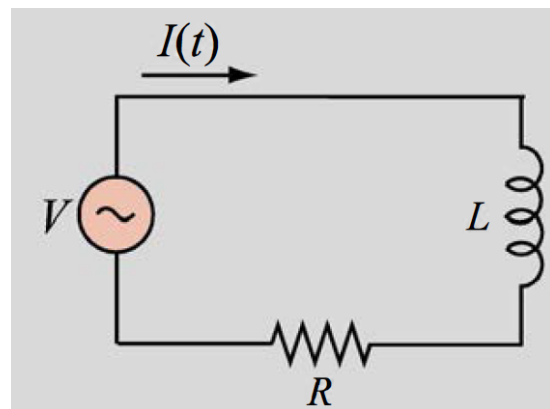
```
>> a=0; b=2.5;
>> h=0.05; yini=3;
>> [x,y]=odeRK4(@ODEFxy,a,b,h,yini);
Solusion Exacta
>> xp=a:0.1:b;
>>yp=70/9*exp(-0.3*xp)-43/9*exp(-1.2*xp);
>>plot(x,y, '*r' ,xp,yp)
```



Problem

An inductor L = 15 H and a resistor R = 1000 ohms are connected in series with an AC power source providing voltage of V = 10sin(2π*v*t) Vots, where *v*= 100 kHz, as shown in the figure. The current *I*(t) in the circuit is determined from the solution of the equation:

$$\frac{dI}{dt} = \frac{10\sin(2\pi \nu t)}{L} - \frac{R}{L}I$$



Solve the equation and plot the current as a function of time for 0 <= t<=1 x 10^{-4} s with *I*(0) = 0. Using h=10^{-9} s.